

Łojasiewicz exponents and Farey sequences

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Abstract

Let I be an ideal of the ring of formal power series $\mathbf{K}[[x, y]]$ with coefficients in an algebraically closed field \mathbf{K} of arbitrary characteristic. Let Φ denote the set of all parametrizations $\varphi = (\varphi_1, \varphi_2) \in \mathbf{K}[[t]]^2$, where $\varphi \neq (0, 0)$ and $\varphi(0, 0) = (0, 0)$. The purpose of this paper is to investigate the invariant

$$\mathcal{L}_0(I) = \sup_{\varphi \in \Phi} \left(\inf_{f \in I} \frac{\text{ord } f \circ \varphi}{\text{ord } \varphi} \right)$$

called the *Łojasiewicz exponent* of I . Our main result states that for the ideals I of finite codimension the Łojasiewicz exponent $\mathcal{L}_0(I)$ is a Farey number i.e. an integer or a rational number of the form $N + \frac{b}{a}$, where a, b, N are integers such that $0 < b < a < N$.

1 Introduction

Let \mathbf{K} be an algebraically closed field of arbitrary characteristic. Let t be a variable. A *parametrization* is a pair $\varphi(t) = (\varphi_1(t), \varphi_2(t)) \in \mathbf{K}[[t]]^2 \setminus \{(0, 0)\}$ such that $\varphi_1(0) = \varphi_2(0) = 0$. We put $\text{ord } \varphi = \inf\{\text{ord } \varphi_1, \text{ord } \varphi_2\}$, where $\text{ord } \varphi_k$ stands for the *order* of the power series $\varphi_k = \varphi_k(t)$. For any ideal $I \subset \mathbf{K}[[x, y]]$ we consider the *Łojasiewicz exponent* $\mathcal{L}_0(I)$ (see [L-T], [P1], [D], [dF], [B-R]) defined by the formula

$$\mathcal{L}_0(I) = \sup_{\varphi \in \Phi} \left(\inf_{f \in I} \frac{\text{ord } f \circ \varphi}{\text{ord } \varphi} \right),$$

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where Φ stands for the set of all parametrizations $\varphi = (\varphi_1, \varphi_2)$.

Note that $\mathcal{L}_0(I) < +\infty$ if and only if I is of finite codimension.

In the framework of the complex analytic geometry the notion of Lojasiewicz exponent was introduced and studied by Lejeune-Jalabert and Teissier [L-T]. They considered much more general notion including the Lojasiewicz exponent of holomorphic ideals in several variables. D'Angelo [D] defined this invariant independently and gave its applications to complex function theory on domains in \mathbf{C}^n . Recently Cassou-Noguès and Veys [CN-V] introduced an algorithm to study ideals in $\mathbf{K}[[x, y]]$ which enables us to compute $\mathcal{L}_0(I)$ using a finite sequence of Newton diagrams.

Let $g \in \mathbf{K}[[x, y]]$ be an irreducible power series. We put

$$\mathcal{L}_0(I, g) = \inf_{f \in I} \left\{ \frac{\text{ord } f \circ \varphi}{\text{ord } \varphi} \right\},$$

where φ is a parametrization such that $g \circ \varphi = 0$. The notion does not depend on the choice of φ . If $\mathcal{L}_0(I) = \mathcal{L}_0(I, g)$ then we say that the Lojasiewicz exponent *is attained on the branch* $g = 0$.

Theorem 1.1 ([B-R, Theorem 6]) *Let $I \subset \mathbf{K}[[x, y]]$ be a proper ideal and let f_1, \dots, f_m be generators of I . Then there is an irreducible factor g of the power series $f_1 \cdots f_m$ such that $\mathcal{L}_0(I)$ is attained on the branch $g = 0$.*

This result was proved by Chądzynski and Krasinski [Ch-K, Theorem 3] and independently by McNeal and Némethi [M-N, Theorem 1.1] for holomorphic ideals. The case of ideals in $\mathbf{K}[[x, y]]$, where \mathbf{K} is of arbitrary characteristic is due to Brzostowski and Rodak [B-R, Theorem 6]. In Section 2 of this note we give a very short proof of it. Let us write down the following corollary to Theorem 1.1.

Corollary 1.2 *If $I \subset \mathbf{K}[[x, y]]$ is of finite codimension then $\mathcal{L}_0(I)$ is a rational number.*

Our main result is

Theorem 1.3 *Let I be an ideal of $\mathbf{K}[[x, y]]$ of finite codimension. Then $\mathcal{L}_0(I)$ is a Farey number i.e. $\mathcal{L}_0(I)$ is an integer or a rational number of the form $N + \frac{b}{a}$, where N, a, b are integers such that $0 < b < a < N$.*

Theorem 1.3 gives a positive answer to Question 1 of [B-R]. It implies that the fractional parts of the Łojasiewicz exponents $\mathcal{L}_0(I)$ form the Farey sequences of order $\lfloor \mathcal{L}_0(I) \rfloor$ (see [H-W]), where $\lfloor z \rfloor$ denotes the integer part of the real number z .

The proof of Theorem 1.3 is given in Section 3.

The holomorphic version (I is a holomorphic ideal generated by two elements) was proved in [P1, Theorem 3.4]. Its proof does not extend to the case of arbitrary characteristic.

2 Proof of Theorem 1.1

For any $f, g \in \mathbf{K}[[x, y]]$ we consider the intersection number $i_0(f, g) = \dim_{\mathbf{K}} \mathbf{K}[[x, y]]/(f, g)$, where (f, g) is the ideal generated by f and g in $\mathbf{K}[[x, y]]$. Let

$$d(f, g) = \frac{i_0(f, g)}{\text{ord } f \text{ ord } g}$$

for irreducible $f, g \in \mathbf{K}[[x, y]]$. Then $d(f, g)$ is a *logarithmic distance* on the set of all irreducible power series, that is

$$(D1) \quad d(f, f) = +\infty,$$

$$(D2) \quad d(f, g) = d(g, f), \text{ and}$$

$$(D3) \quad d(f, g) \geq \inf\{d(f, h), d(g, h)\} \text{ for } f, g, h \text{ irreducible power series.}$$

Only Property (D3) is non-trivial (see [GB-P, Corollary 2.9]).

If $g \in \mathbf{K}[[x, y]]$ is irreducible then there exists a parametrization $\psi^o \in \mathbf{K}[[t]]^2$ such that $g \circ \psi^o = 0$ and $\text{ord } \psi^o = \text{ord } g$. Moreover, for any power series $f \in \mathbf{K}[[x, y]]$ we have $i_0(f, g) = \text{ord}(f \circ \psi^o)$. If ψ is a parametrization such that $g \circ \psi = 0$ then there exists $\tau \in \mathbf{K}[[t]]$ of positive order such that $\psi = \psi^o \circ \tau$. The equality $\frac{\text{ord}(f \circ \psi)}{\text{ord } \psi} = \frac{\text{ord}(f \circ \psi^o)}{\text{ord } \psi^o} = \frac{i_0(f, g)}{\text{ord } g}$ shows that the definition

of $\mathcal{L}_0(I, g)$ is correct and can be rewritten as follows $\mathcal{L}_0(I, g) = \inf_{f \in I} \frac{i_0(f, g)}{\text{ord } g}$.

If φ is a parametrization then there exists an irreducible power series $g \in \mathbf{K}[[x, y]]$ such that $g \circ \varphi = 0$. This shows that

$$\mathcal{L}_0(I) = \sup \{ \mathcal{L}_0(I, g) : g \text{ is irreducible} \}.$$

If $I = (f_1, \dots, f_m)$ then

$$\mathcal{L}_0(I, g) = \inf_{1 \leq k \leq m} \frac{i_0(f_k, g)}{\text{ord } g}. \quad (1)$$

Lemma 2.1 *Let $I = (f_1, \dots, f_m)$ and let $\prod_i f_i = \prod_j h_j$ with $h_j \in \mathbf{K}[[x, y]]$ irreducible. Let $g \in \mathbf{K}[[x, y]]$ be an irreducible power series. Take an index k such that $d(g, h_k) = \sup_j \{d(g, h_j)\}$. Then $\mathcal{L}_0(I, g) \leq \mathcal{L}_0(I, h_k)$.*

Proof. Let us denote h_k by h . Then $d(g, h) \geq d(g, h_j)$ for any index j . After (D3) we get $d(h, h_j) \geq \inf\{d(g, h), d(g, h_j)\} = d(g, h_j)$. Therefore for any j we have $\frac{i_0(g, h_j)}{\text{ord } g} \leq \frac{i_0(h, h_j)}{\text{ord } h}$ and consequently for any $i \in \{1, \dots, m\}$ we get $\frac{i_0(g, f_i)}{\text{ord } g} \leq \frac{i_0(h, f_i)}{\text{ord } h}$, which implies $\mathcal{L}_0(I, g) \leq \mathcal{L}_0(I, h)$. ■

Now, we can prove Theorem 1.1.

Proof of Theorem 1.1. We keep the notations of Lemma 2.1. Fix an irreducible power series g . Then $\mathcal{L}_0(I, g) \leq \mathcal{L}_0(I, h_k)$. Hence $\mathcal{L}_0(I) \leq \sup_j \{\mathcal{L}_0(I, h_j)\}$. The inverse inequality is obvious. Therefore $\mathcal{L}_0(I) = \sup_j \{\mathcal{L}_0(I, h_j)\}$, which proves Theorem 1.1. ■

3 Proof of Theorem 1.3

Let $f = \sum c_{\alpha\beta} x^\alpha y^\beta \in \mathbf{K}[[x, y]]$. The *Newton diagram* $\Delta(f)$ of f is by definition the convex hull of the set $\{(\alpha, \beta) \in \mathbf{N}^2 : c_{\alpha\beta} \neq 0\} + \mathbf{R}_{\geq 0}^2$. We use Teissier's notation ([L-T, p. 846]) denoting by $\left\{\frac{b}{a}\right\}$ the Newton diagram of $y^a + x^b$, for $a, b > 0$. The following properties of Newton diagrams are well-known

(N1) for generic c_1, \dots, c_m : $\Delta(\sum_{i=1}^m c_i f_i)$ is the convex hull of the set $\bigcup_{i=1}^m \Delta(f_i)$,

(N2) if $f = 0$ is a branch different from the axes then $\Delta(f) = \left\{\frac{i_0(f, y)}{i_0(f, x)}\right\}$,

(N3) if $\Delta(f_1) = \left\{\frac{b_1}{a_1}\right\}$ and $\Delta(f_2) = \left\{\frac{b_2}{a_2}\right\}$ then $i_0(f_1, f_2) \geq \min\{a_1 b_2, a_2 b_1\}$,
with equality if $a_1 b_2 \neq a_2 b_1$.

Property (N1) is a consequence of the definition of $\Delta(f)$. For Property (N2) see [P2, Proposition 4.2]. Property (N3) follows from [P2, Proposition 3.13, Proposition 3.8 (v)].

Let I be an ideal of $\mathbf{K}[[x, y]]$ with a finite Łojasiewicz exponent. Put $l = \mathcal{L}_0(I)$. Consider the set of ideals $J \subset \mathbf{K}[[x, y]]$ such that $\mathcal{L}_0(J) = l$ and let M be a maximal element of this set (with respect to the inclusion). Set $\text{ord } M = \inf\{\text{ord } f : f \in M\}$. Observe that replacing any finite sequence of

generators of M by their general linear combinations we obtain generators of the same order, equal to $\text{ord } M$.

Lemma 3.1 *If f_1, \dots, f_m is a sequence of generators of M of the same order then there exists $k \in \{1, \dots, m\}$, such that f_k is irreducible and $\mathcal{L}_0(M, f_k) = \mathcal{L}_0(M)$.*

Proof. Let f_1, \dots, f_m be a sequence of generators of M of the same order. By Theorem 1.1 the Łojasiewicz exponent of M is attained on an irreducible factor h of the product $f_1 \cdots f_m$.

Let \overline{M} be the ideal generated by f_1, \dots, f_m and h . Since $M \subset \overline{M}$ we get $\mathcal{L}_0(M) \geq \mathcal{L}_0(\overline{M})$. On the other hand $\mathcal{L}_0(\overline{M}) \geq \mathcal{L}_0(\overline{M}, h) = \mathcal{L}_0(M, h) = \mathcal{L}_0(M)$, which implies $\mathcal{L}_0(\overline{M}) = \mathcal{L}_0(M)$. By the maximality of M we get $h \in M$. Let $k \in \{1, \dots, m\}$ be an index such that h divides f_k . Then $\text{ord } h \leq \text{ord } f_k = \text{ord } M \leq \text{ord } h$, hence $\text{ord } h = \text{ord } f_k$ and $f_k = h \cdot u$, where $u(0, 0) \neq 0$ which implies that f_k is irreducible. ■

Let us pass to the proof of Theorem 1.3.

Proof of Theorem 1.3. We keep the notation and assumptions introduced above. It suffices to prove that $\mathcal{L}_0(M)$ is an integer or $\mathcal{L}_0(M) = N + \frac{b}{a}$, where $0 < b < a < N$.

By Lemma 3.1 there exists an irreducible power series $h \in M$ of order $\text{ord } h = \text{ord } M$ such that $\mathcal{L}_0(M, h) = \mathcal{L}_0(M)$.

If $\text{ord } h = 1$ then $\mathcal{L}_0(M)$ is an integer.

If $\text{ord } h > 1$ then changing the system of coordinates if necessary, we may assume that $\text{ord } h(0, y) < \text{ord } h(x, 0)$. Let $a = \text{ord } h(0, y)$ and $c = \text{ord } h(x, 0)$. Then $\Delta(h) = \left\{ \frac{c}{a} \right\}$, where $a = \text{ord } h = \text{ord } M$.

Replacing any finite sequence of generators of M by a sequence of their linear generic combinations we get a sequence f_1, \dots, f_m of generators of the same order such that $\Delta(f_1) = \dots = \Delta(f_m)$. Let Δ be their common Newton diagram.

Since $h \in M$, we have $h = a_1 f_1 + \dots + a_m f_m$, where $a_i \in \mathbf{K}[[x, y]]$. Substituting $x = 0$ we get $\text{ord } M = \text{ord } h(0, y) \geq \min_i \{\text{ord } f_i(0, y)\} \geq \text{ord } M$. Hence the diagram Δ intersects the vertical axis at $(0, a)$. By Lemma 3.1 at least one of f_1, \dots, f_m is irreducible. This implies that Δ has only one compact face. Since $\text{ord } f_i = a$, we have $\Delta = \left\{ \frac{d}{a} \right\}$, where $d \geq a$.

By (1) there is $k \in \{1, \dots, m\}$ such $\mathcal{L}_0(M) = \frac{i_0(f_k, h)}{\text{ord } h}$.

If $d = a$ then by (N3) we get $i_0(f_k, h) = \min\{ac, a^2\} = a^2$. In this case $\mathcal{L}_0(M) = a$.

If $d > a$ then by (N3) we get $i_0(f_k, h) \geq \min\{ac, ad\} = a \min\{c, d\} \geq a(a+1)$. Write $i_0(f_k, h) = aN + b$, where $0 \leq b < a$. Dividing this equality by a and taking integer parts we get $N = \left\lfloor \frac{i_0(f_k, h)}{a} \right\rfloor \geq \frac{a(a+1)}{a} = a+1$. Therefore $\mathcal{L}_0(M) = N + \frac{b}{a}$, where $0 \leq b < a < N$, which completes the proof. ■

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